

# IMPORTANCE SAMPLING FOR A MARKOVIAN INTENSITY MODEL WITH APPLICATIONS TO CREDIT RISK

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**ABSTRACT.** This paper considers importance sampling for estimation of rare-event probabilities in a Markovian intensity model commonly used in the context of credit risk. The main contribution is the design of efficient importance sampling algorithms using subsolutions of a certain Hamilton-Jacobi equation. We provide theoretical results that quantify the performance of importance sampling algorithms and for certain instances of the model under consideration the proposed algorithm is proved to be asymptotically optimal. The computational gain compared to standard Monte Carlo is illustrated by numerical examples.

## 1. INTRODUCTION

Events of the past decade have made it abundantly clear that rare events are of particular importance in the financial context due to the catastrophic impact they may have for, say, a company or a financial institution. As the mathematical models involved are becoming increasingly complex, combined with the need for fast and accurate results, the need for efficient simulation algorithms has grown as well. The purpose of this paper is to consider a particular class of algorithms, namely importance sampling, in the context of a Markovian intensity model for credit risk.

In the recent papers [2, 3] different types of Monte Carlo methods are studied for the specific problem of estimating rare-event probabilities in two types of models for credit risk. In [2] the number of defaults in a portfolio is described by a Markovian intensity model and in [3] the authors consider a (discrete-time version of a) first-passage model, based on a structural model with stochastic volatility. In both papers the authors study the performance of Monte Carlo methods for the task of estimating rare events, here characterized by large portfolio losses. Particularly relevant for our work is that in [2] both importance sampling and an interacting particle system (IPS) approach are used and the constructed importance sampling schemes show unsatisfactory performance for certain choices of parameters in the underlying model.

This paper studies further the design of importance sampling algorithms for the type of Markovian intensity models used in [2]. The task is to estimate the probability that the number of defaults in a portfolio exceeds some threshold before a fixed time horizon. The problem fits into the more general context of finding suitable sampling distributions for a certain type of Markovian birth processes

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with Markovian intensity, as much of the analysis does not depend on the specific form of the intensity. The objective of the paper is to illustrate that the so-called subsolution approach [12] can be applied and yield satisfactory results in a wide range of situations. In particular, recent results on the construction of subsolutions to Hamilton-Jacobi equations [6] provide a coherent framework for the construction of efficient sampling distributions. However, the paper is not meant as a general discussion of the merits of importance sampling vs. IPS methods, but rather to showcase the usefulness of the subsolution approach for situations in which finding a suitable change-of-measure is not a trivial task.

Although the motivation comes from the credit risk setting, and in particular the paper [2], the work can be viewed in the more general context of Monte Carlo methods for Markovian intensity models with mean-field characteristics. In terms of modeling credit risk and defaults in large portfolios there has been much work in recent years. For example, in [21, 22, 23] the authors consider more advanced models for the stochastic default intensity, meant to capture properties observed in the market, and study the behavior of defaults as the size of the portfolio goes to infinity. One of the objectives is to use the limiting behavior as an approximation for finite but large portfolios. In [21] some numerical experiments are conducted and the authors specifically remark that standard Monte Carlo typically is slow for the large portfolios and long time horizons that one typically encounters (hence their desire to develop new methods). Thus, understanding how to design efficient Monte Carlo methods, in this case importance sampling, even for rather simple models, is a valuable step towards being able to construct fast and accurate numerical methods for the more involved systems. The latter requires further insights into the design of importance sampling algorithms for interacting particle systems.

In order to design efficient importance sampling algorithms for the model under consideration we use the subsolution approach by Dupuis and Wang, combined with recent results on representations of viscosity solutions of Hamilton-Jacobi equations [6]. To the best of our knowledge this is the first work to apply this technique for designing importance sampling algorithms to the setting of a pure-jump process and state dynamics that do not change only across certain boundaries in the state space; compare for example to the queueing model in [8]. The qualitative difference is that in the current setting affine functions of the state will not produce efficient algorithms but the gradient of the subsolution must also be state-dependent. This also explains why the algorithms used in [2] show poor performance.

The connection between subsolutions of Hamilton-Jacobi equations and importance sampling, first encountered in [11] and more extensively developed in [12], has been used to construct efficient importance sampling algorithms in a number of different models, particularly for queueing systems [8, 9, 13], but also in the diffusion setting [10]. This technique has also been used in contexts other than importance sampling, for example splitting algorithms [4]. For a general overview of Monte Carlo methods used in financial engineering the monograph [16] is an excellent source; examples of the use of importance sampling can be found in [17, 18, 19].

The paper is organized as follows. In Section 2 we introduce the Markovian intensity model for credit risk, associated stochastic processes and probabilities of interest. Section 3 reviews large deviation results for the type of Markov processes used to model the number of defaults in a portfolio. Importance sampling, particularly for the type of Markov processes defined in Section 2, and the relevant

measure of efficiency is discussed in Section 4. In Section 5 we discuss the notion of subsolutions, the connection between efficient importance sampling and Hamilton-Jacobi (HJ) equations and construct subsolutions to the HJ equation associated with the model under consideration. In Section 6 theoretical results on the performance of importance sampling algorithms based on subsolutions are proved. In particular, asymptotic optimality for the proposed algorithm, for certain choices of parameter values in the credit risk model, follow as a corollary. Lastly, in Section 7 numerical experiments are presented that illustrate the performance of the proposed importance sampling algorithms. For completeness, the Appendix contains a formal derivation of the HJ equation of Section 5.

## 2. MODEL AND PROBLEM FORMULATION

Consider a population of  $n \in \mathbb{N}$  individuals divided into  $d$  homogeneous groups, with  $w_j n$  individuals in the  $j$ th group. In the context of credit risk we think of a credit portfolio and the  $n$  individuals are the obligors in that portfolio. It is assumed that  $w_j > 0$  for each  $j = 1, \dots, d$ , and  $w_1 + \dots + w_d = 1$ . For notational convenience, define  $\Omega$  to be the set

$$\Omega = \prod_{j=1}^d [0, w_j].$$

This will act as the state space for the stochastic processes we consider. Let  $\{Q^n(t); t \geq 0\}$ ,  $Q^n(0) = 0$ , denote a  $d$ -dimensional continuous-time pure jump Markov process, where  $Q_j^n(t)$  represents the number of defaults in the  $j$ th group up to time  $t$ . Let  $\lambda : \Omega \rightarrow [0, \infty)^d$  be a continuous function and take the jump intensity of the process  $Q^n$  from state  $nx$  to state  $nx + e_j$  to be

$$r^n(x; e_j) = n\lambda_j(x).$$

The total jump intensity, when in state  $nx$ , is denoted by  $R(x)$ ,

$$R(x) = \sum_{j=1}^d r^n(x; e_j) = \sum_{j=1}^d n\lambda_j(x).$$

In all examples  $\lambda$  will be of the form

$$\lambda_j(x) = a_j(w_j - x_j)e^{b\langle 1, x \rangle}, \quad j = 1, \dots, d, \quad (2.1)$$

for  $a_1, \dots, a_d$  and  $b$  in  $\mathbb{R}_+$ . The vector  $a = (a_1, \dots, a_d)$  is supposed to reflect the default intensities in the  $d$  different homogeneous groups whereas  $b$  determines the contagion effect of the total number of defaults on the entire portfolio. The model (2.1) is a minor generalization of that used in the examples in [2], in that different groups are allowed to have different intensities  $a_i$ . In [2], the authors hint at a model of the form (2.1), with inhomogeneous groups, but never explicitly state or consider any such examples.

Let  $T_1, T_2, \dots$  be the jump times of  $X^n$ ,  $T_0 = 0$ , and  $\tau_k = T_k - T_{k-1}$  the time between jumps,  $k = 1, 2, \dots$ . The stochastic kernel of  $Q^n$  is then given by

$$\begin{aligned} \Theta^n(dt, e_j \mid x) &= \mathbb{P}(\tau_{k+1} = dt, Q^n(T_{k+1}) - Q^n(T_k) = e_j \mid Q^n(T_k) = nx) \\ &= r^n(x; e_j)e^{-R(x)t}dt, \end{aligned} \quad (2.2)$$

where  $k$  is some integer.

We are interested in studying the probability of the process  $Q^n$  exceeding some (high) threshold before time  $T$ . More precisely, for some  $z \in (0, 1)$ , we study the probability  $p_n$  given by

$$p_n = \mathbb{P}\left(\sum_{j=1}^d Q_j^n(T) \geq nz\right) = \mathbb{P}\left(\sum_{j=1}^d Q_j^n(t) \geq nz, \text{ for some } t \leq T\right).$$

The second equality follows from the fact that  $Q^n$  is non-decreasing. For large  $n$  the event that  $Q^n$  exceeds  $nz$  before time  $T$  is a rare event, i.e., the probability  $p_n$  will be small. For such  $z$  standard Monte Carlo will be inefficient for estimating  $p_n$  and our goal is to construct efficient importance sampling algorithms for this task. Difficulties arise when the intensity  $\lambda$  is state-dependent (see e.g. (2.1) below), requiring a detailed analysis for the design of the sampling distribution.

In the context of credit risk, the described problem amounts to studying the probability of a large number of defaults in a group of  $n$  obligors. Specifically, the probability that the number of defaults exceeds  $nz$  before time  $T$ .

The idea is to use asymptotic results as  $n$  goes to infinity to guide the design of importance sampling algorithms. Denote by  $\{X^n(t); t \geq 0\}$  the scaled process

$$X^n(t) = \frac{1}{n} Q^n(t). \quad (2.3)$$

The probability  $p_n$  can then be expressed in terms of the scaled process  $X^n$ ,

$$p_n = \mathbb{P}\left(\sum_{j=1}^d X_j^n(T) \geq z\right) = \mathbb{P}\left(\sum_{j=1}^d X_j^n(t) \geq z \text{ for some } t \leq T\right), \quad (2.4)$$

and we use the asymptotics for this probability (i.e., for the process  $X^n$ ) to aid in the choice of sampling distribution.

### 3. LARGE DEVIATIONS FOR THE SEQUENCE OF SCALED JUMP-PROCESSES

The asymptotics eluded to in Section 2 are the large deviation asymptotics associated with  $X^n$  as  $n$  goes to infinity. Here, for each  $n$ , the process  $\{X^n(t); t \geq 0\}$  in (2.3) is a continuous-time pure jump Markov process with infinitesimal generator  $\mathcal{A}^n$  defined by

$$\mathcal{A}^n f(x) = n \sum_{j=1}^d \lambda_j(x) [f(x + e_j/n) - f(x)],$$

for some suitable class of functions  $f$ . These processes take values (with probability one) in the space  $\mathcal{D}([0, \infty); \mathbb{R}^d)$  of càdlàg functions from  $[0, \infty)$  to  $\mathbb{R}^d$ . To the generator  $\mathcal{A}^n$  there is an associated scaled Hamiltonian,  $H^n$ , defined by

$$H^n f(x) = \frac{1}{n} e^{-nf(x)} \mathcal{A}^n e^{nf(x)} = \sum_{j=1}^d \lambda_j(x) (e^{n(f(x+e_j/n)-f(x))} - 1).$$

If, for example, the function  $f$  is  $C^1$  and the sum  $\sum_j \lambda_j(x) e^{\langle \alpha, e_j \rangle}$  is finite for  $\alpha \in \mathbb{R}^d$ , it holds that

$$\lim_{n \rightarrow \infty} H^n f(x) = \sum_{j=1}^d \lambda_j(x) (e^{\langle Df(x), e_j \rangle} - 1).$$

Define the function  $H : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$H(x, \alpha) = \sum_{j=1}^d \lambda_j(x) (e^{\langle \alpha, e_j \rangle} - 1), \quad (3.1)$$

and let  $L$  be the convex conjugate of  $H$ ,

$$L(x, \beta) = \sup_{\alpha \in \mathbb{R}^d} [\langle \alpha, \beta \rangle - H(x, \alpha)].$$

A straightforward calculation gives the explicit form of  $L$ ,

$$L(x, \beta) = \langle \beta, \log \frac{\beta}{\lambda(x)} \rangle - \langle \beta - \lambda(x), 1 \rangle,$$

where  $\beta/\lambda(x)$  denotes component-wise division.

For any  $T \in [0, \infty)$ , let  $\mathcal{AC}([0, T]; \mathbb{R}^d)$  be the set of all absolutely continuous functions  $\psi : [0, T] \rightarrow \mathbb{R}^d$  and define the process-level rate function by

$$I_x(\psi) = \begin{cases} \int_0^T L(\psi(t), \dot{\psi}(t)) dt, & \psi \in \mathcal{AC}([0, T]; \mathbb{R}^d), \text{ non-decreasing and } \psi(0) = x, \\ \infty, & \text{otherwise.} \end{cases}$$

For each  $T < \infty$ , the sequence  $\{X^n\}$  satisfies the following Laplace principle on the sample path level; see [14, 24]. Note that, because  $\mathcal{D}([0, T]; \mathbb{R}^d)$  is a separable completely metrizable space, this Laplace principle is equivalent to the large deviation principle [5].

**Theorem 3.1.** *The sequence  $\{X^n(t); t \in [0, T]\}$  satisfies the Laplace principle with rate function  $I_x$ : For any  $x \in \mathbb{R}^d$  and bounded, continuous function  $h : \mathcal{D}([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_x[\exp\{-nh(X^n)\}] = -\inf\{I_x(\psi) + h(\psi)\}.$$

We end this section by hinting at how the large deviation principle, guaranteed by Theorem 3.1, connects to the design of efficient simulation algorithms. Let  $D_z$  be the set

$$D_z = \left\{ y \in \Omega : \sum_{j=1}^d y_j \geq z \right\},$$

and define the function  $U : [0, T] \times \Omega \rightarrow [0, \infty]$  by

$$U(t, x) = \inf_{\psi} \left\{ \int_t^T L(\psi(s), \dot{\psi}(s)) ds : \psi(t) = x, \psi(T) \in D_z \right\}, \quad (3.2)$$

where the infimum is over  $\psi \in \mathcal{AC}([0, T]; \mathbb{R}^d)$  that are non-negative and non-decreasing. For each pair  $(t, x)$ ,  $U(t, x)$  is interpreted as the large deviation rate of the probability of reaching the set  $D_z$  before time  $T$ , when starting in state  $x$  at time  $t$ . According to Theorem 3.1, the convex conjugate  $L$  of  $H$  acts as the local rate function for the sequence  $\{X^n\}$ . It is this conjugacy between  $L$  and  $H$  that provides the connection to a Hamilton-Jacobi equation, which can be used for designing efficient simulation algorithms; see Section 5 and the Appendix.

## 4. IMPORTANCE SAMPLING

Before we embark on the task of constructing sampling schemes for the type of Markovian intensity model described in Section 2, we first review the basics of importance sampling and the relevant measure of efficiency for estimating rare-event probabilities such as (2.4). We also describe how to construct importance sampling distributions for  $X^n$ , defined in (2.3). That is, we give the form of the change-of-measure and the definition of the importance sampling estimator of (2.4). For a more thorough introduction to importance sampling see, e.g., [1, 20].

**4.1. Basics of importance sampling.** Importance sampling is the method to simulate a system under different dynamics, i.e., probability distribution, than in the original model. In the present setting the task is to estimate the probability  $p_n = \mathbb{P}(X^n(T) \in D_z)$ , where  $\mathbb{P}$  describes the original dynamics for the process  $X^n$ . To perform importance sampling, consider different dynamics and the associated probability measure  $\bar{\mathbb{Q}}^n$ ,  $\mathbb{P} \ll \bar{\mathbb{Q}}^n$  (on an appropriate part of the state space). One sample of the importance sampling estimator, denoted by  $\hat{p}_n$ , is the indicator of the event times the Radon-Nikodym derivative associated with the change of measure from  $\mathbb{P}$  to  $\bar{\mathbb{Q}}^n$ ,

$$\hat{p}_n = I\{X^n(T) \in D_z\} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n},$$

where  $X^n$  now has dynamics according to  $\bar{\mathbb{Q}}^n$ . Including the Radon-Nikodym derivative ensures that  $\hat{p}_n$  is an unbiased estimator of  $p_n$ ,

$$\mathbb{E}_{\bar{\mathbb{Q}}^n}[\hat{p}_n] = \mathbb{E}_{\bar{\mathbb{Q}}^n} \left[ I\{X^n(T) \in D_z\} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} \right] = \mathbb{E}_{\mathbb{P}}[I\{X^n(T) \in D_z\}] = p_n.$$

To choose what alternative measure  $\bar{\mathbb{Q}}^n$  to use we need a measure of efficiency. Unbiasedness of the estimator  $\hat{p}_n$  suggests that efficiency can be measured in terms of the second moment of  $\hat{p}_n$ ; a smaller second moment corresponds to a more efficient algorithm. Hence, the aim is to choose a sampling distribution that minimizes this second moment with respect to the sampling distribution, whilst still being feasible to implement (cf. the optimal zero-variance change of measure [1]).

How small can we hope for the second moment to be? The exponential rate of decay of  $p_n$  is governed by the large deviation principle of Theorem 3.1. By Jensen's inequality,  $\mathbb{E}_{\bar{\mathbb{Q}}^n}[\hat{p}_n^2] \geq p_n^2$  and it follows that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\bar{\mathbb{Q}}^n}[\hat{p}_n^2] \geq 2 \liminf_{n \rightarrow \infty} \frac{1}{n} \log p_n = -2U(0, 0),$$

where  $U$  is defined in (3.2). This lower bound for the logarithmic asymptotics of  $\hat{p}_n^2$  holds true for any sampling distribution  $\bar{\mathbb{Q}}^n$ . A particular choice is said to be *asymptotically optimal* if the corresponding upper bound holds as well, that is if for that  $\bar{\mathbb{Q}}^n$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\bar{\mathbb{Q}}^n}[\hat{p}_n^2] \leq -2U(0, 0).$$

It is useful to note that, because we are interested in probabilities, the second moment of  $\hat{p}_n$  under  $\bar{\mathbb{Q}}^n$  is equal to the first moment of  $\hat{p}_n$  under  $\mathbb{P}$ ,

$$\mathbb{E}_{\bar{\mathbb{Q}}^n}[\hat{p}_n^2] = \mathbb{E}_{\bar{\mathbb{Q}}^n} \left[ I\{X^n(T) \in D_z\} \left( \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} \right)^2 \right] = \mathbb{E}_{\mathbb{P}} \left[ I\{X^n(T) \in D_z\} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} \right].$$

**4.2. Importance sampling for the process  $X^n$ .** The dynamics of the process  $X^n$  are determined by the stochastic kernel  $\Theta^n$  given in (2.2). For importance sampling, choose a different stochastic kernel  $\bar{\Theta}^n$ ,

$$\bar{\Theta}^n(dt, e_j | x) = \bar{r}^n(x; e_j) e^{-\bar{R}(x)t} dt, \quad (4.1)$$

with  $\bar{R}(x) = \sum_{j=1}^d \bar{r}^n(x; e_j)$ . Similar to  $r^n(x, \cdot)$ , the jump intensities  $\bar{r}^n(x, \cdot)$  are of the form

$$\bar{r}^n(x, e_j) = n \bar{\lambda}_j(x),$$

for some vector  $\bar{\lambda}(x) = (\bar{\lambda}_1(x), \dots, \bar{\lambda}_d(x))$ . That is, just as for  $\lambda$ ,  $\bar{\lambda}$  is a function from  $\Omega$  to  $[0, \infty)^d$  and the jump intensities  $\bar{r}^n(x, \cdot)$  are obtained by scaling this function by  $n$ . Hence, the choice of stochastic kernel  $\bar{\Theta}^n$  is determined by the choice of  $\bar{\lambda}$ .

For  $z \in (0, 1)$ , define  $N^z$  as the number of jumps required for the process to reach the target set  $D_z$ ,

$$N^z = \inf \left\{ k \geq 1 : \sum_{j=1}^d Q_j^n(T_k) \geq nz \right\} = \inf \left\{ k \geq 1 : X^n(T_k) \in D_z \right\},$$

and  $N^0$  as the number of jumps needed to exceed time  $T$ ,

$$N^0 = \inf \{ k \geq 1 : T_k > T \}.$$

A single sample of the importance sampling estimator based on  $\bar{\Theta}^n$  is

$$\hat{p}_n = I\{N^z < N^0\} \prod_{k=1}^{N^n} \frac{\Theta^n(d\tau_k, v_k | X^n(T_{k-1}))}{\bar{\Theta}^n(d\tau_k, v_k | X^n(T_{k-1}))}, \quad (4.2)$$

where  $v_k \in \{e_1, \dots, e_d\}$  is the direction of the  $k$ th jump and the  $\tau_k$ 's denote times between jumps (see Section 2). How to choose the stochastic kernel  $\bar{\Theta}^n$  is the topic of the next section.

## 5. SUBSOLUTIONS AND ASSOCIATED SAMPLING ALGORITHMS

In this section we discuss the role that so-called subsolutions play in the design of efficient Monte Carlo methods. Indeed, as was briefly mentioned in the Introduction, it turns out that efficient Monte Carlo methods, particularly importance sampling, are inherently connected to certain partial differential equations associated with the dynamics of the stochastic system under consideration. The first results in this direction were [11] in which the connection between importance sampling and so-called Isaacs equations was first made. In [12] the results were extended and the connection to subsolutions of the relevant PDEs, which are of Hamilton-Jacobi type, was established. The paper [12] is recommended for the reader who wants an overview of the role subsolutions to Hamilton-Jacobi equations play in the context of importance sampling. In what follows we state the relevant Hamilton-Jacobi equation, define what a subsolution is and introduce a certain type of subsolution that is used to construct efficient importance sampling algorithms for the model described in Section 2. Several ideas and results presented in this section are valid in the more general context of construction of viscosity subsolutions of Hamilton-Jacobi equations. For the more general results and details we refer to the paper [6]; here we only include the parts that are relevant for the specific model under consideration.

**5.1. Hamilton-Jacobi equation and choice of sampling distribution for Markovian intensity models.** We start by motivating the form of the Hamilton-Jacobi equation. The current setting is similar to that of the queueing model in [8], the main differences being the state dynamics and the fact that time now plays a role in the estimation problem. For completeness, the Isaacs equation is derived in some detail in the Appendix and rigorous proofs on performance for importance sampling algorithms based on subsolutions to the Hamilton-Jacobi equation are given in Section 6.

From the discussion in Section 4, to optimize performance it is desirable to minimize the second moment of  $\hat{p}_n$ , defined in (4.2), among transition kernels  $\bar{\Theta}^n$  described by (4.1). To this end, consider the value function

$$V^n = \inf_{\bar{\Theta}^n} \mathbb{E}_{\bar{\Theta}^n} \left[ I\{N^z < N^0\} \prod_{k=1}^{N^z} \frac{\Theta^n(d\tau_k, v_k \mid X^n(T_{k-1}))}{\bar{\Theta}^n(d\tau_k, v_k \mid X^n(T_{k-1}))} \right],$$

and an associated large deviation type scaling,

$$W^n = -\frac{1}{n} \log V^n.$$

The scaled value function  $W^n$  can be analyzed by considering an associated stochastic control problem. A first step in this direction is to find a dynamic programming equation for  $V^n$ . Define  $V^n(t, x)$  by

$$V^n(t, x) = \inf_{\bar{\Theta}^n} \mathbb{E}_{\bar{\Theta}^n} \left[ I\{N^z < N^0\} \prod_{k=l}^{N^z} \frac{\Theta^n(d\tau_k, v_k \mid X^n(T_{k-1}))}{\bar{\Theta}^n(d\tau_k, v_k \mid X^n(T_{k-1}))} \mid X^n(t) = x \right], \quad (5.1)$$

where  $l$  is such that  $T_{l-1} \leq t < T_l$ .  $W^n(t, x)$  is defined analogously to  $W^n$ . It is clear that  $V^n$  satisfies the terminal condition

$$V^n(T, x) = \begin{cases} 1, & x \in D_z \\ 0, & \text{otherwise,} \end{cases}$$

which for  $W^n$  translates to

$$W^n(T, x) = \begin{cases} 0, & x \in D_z \\ \infty, & \text{otherwise.} \end{cases}$$

Using an analysis reminiscent of the weak convergence approach to large deviations [7], we can formally argue that as  $n$  goes to infinity  $W^n$  converges to  $W$ , the solution of

$$\begin{cases} W_t(t, x) - 2H\left(x, -\frac{DW(t, x)}{2}\right) = 0 & (t, x) \in [0, T) \times \Omega \setminus D_z, \\ W(T, x) = 0, & x \in D_z. \end{cases} \quad (5.2)$$

The details of this (formal) argument are provided in the Appendix. Note that since rigorous results on performance for algorithms proposed from equation (5.2) are provided (Section 6), the derivation of the equation is in itself not needed.

As discussed in the first paragraph of this section, and thoroughly explained in [12], for the purpose of constructing efficient importance sampling algorithms, it is enough to consider subsolutions of (5.2). A classical subsolution of (5.2) is a continuously differentiable function  $\bar{W}$  that satisfies

$$\bar{W}_t(t, x) - 2H\left(x, -\frac{D\bar{W}(t, x)}{2}\right) \geq 0, \quad (t, x) \in [0, T) \times (\Omega \setminus D_z), \quad (5.3)$$



and

$$\bar{W}(T, x) \leq 0, \quad x \in D_z. \quad (5.4)$$

A more general definition is available in terms of viscosity solutions, see [6] and the references therein.

Suppose that  $\bar{W}$  is a subsolution to (5.2). The formal derivation of the Isaacs equation suggests that the sampling distribution  $\mathbb{Q}^n$  should be constructed from  $\bar{W}$  by using jump intensities

$$\bar{\lambda}_j(x) = \lambda_j(x) \exp \left\{ -\frac{\langle D\bar{W}(t, x), e_j \rangle}{2} \right\}, \quad j = 1, \dots, d. \quad (5.5)$$

This is the form used for sampling distributions throughout the remainder of this paper. A particular result from [12] is that, for rather general models, the performance of an importance sampling algorithm based on a subsolution  $\bar{W}$  is determined by the initial value  $\bar{W}(0, 0)$ . The corresponding result for the model under consideration here is proved in Section 6.

As a comparison we mention here that in [2] the choice of jump intensities are of the form

$$\bar{\lambda}_j(x) = \alpha \lambda_j(x),$$

for different values of  $\alpha$ . This choice of sampling distribution corresponds to a state-independent change of measure. Such jump intensities are obtained as a special case of (5.5) by considering affine (in  $x$ ) subsolutions. However, such affine subsolutions will typically not have a maximal initial value, which explains the poor performance observed in [2].

Before describing an explicit construction of subsolutions to (5.2), we end this subsection with a comment on the connection between the function  $U$  defined in (3.2) and the Hamilton-Jacobi equation (5.2). Recall that  $U(t, x)$  is the variational representation of the large deviation rate of the probability of reaching the set  $D_z$  before time  $T$ , starting in  $x$  at time  $t$ . It turns out that the function  $U(t, x)$  is a viscosity solution to the evolutionary Hamilton-Jacobi equation

$$\begin{cases} U_t(t, x) - H(x, -DU(t, x)) = 0, & (t, x) \in [0, T) \times (\Omega \setminus D_z), \\ U(T, x) = 0, & x \in D_z. \end{cases} \quad (5.6)$$

This is a well-known fact; a rigorous proof is provided in [6]. Moreover, subsolutions to the equation (5.6) give rise to subsolutions to the Isaacs equation (5.2). Indeed, a subsolution  $\bar{U}$  to (5.6) satisfies

$$2\bar{U}_t(t, x) - 2H\left(x, -\frac{2D\bar{U}(t, x)}{2}\right) = 2(\bar{U}_t(t, x) - H(x, -D\bar{U}(t, x))) \geq 0,$$

where the inequality follows from the subsolution property. Hence,  $2\bar{U}$  is a subsolution to the Isaacs equation (5.2) and to construct efficient sampling algorithms it suffices to consider subsolutions to the Hamilton-Jacobi equation (5.6). This also implies that if the function  $U$  can be computed explicitly, then one can construct asymptotically optimal importance sampling algorithms by using  $\bar{W} = 2U$  in (5.5).

**5.2. The optimal time-homogeneous sampling distribution.** As an illustration of the general construction of subsolutions described in [6], start by considering the case  $d = 1$  and only algorithms for which the change of measure is independent of  $t$ . That is, if  $\bar{W}$  is the subsolution from which the sampling distribution is constructed, then  $D\bar{W}(t, x)$  is a function of only  $x$ . To emphasize this and to ease notation, let  $\alpha(x) = -D\bar{W}(t, x)/2$ . Thus, the algorithm is based on the jump intensity

$$\bar{\lambda}(x) = \lambda(x) \exp \left\{ -\frac{D\bar{W}(t, x)}{2} \right\} = \lambda(x) \exp \{ \alpha(x) \}.$$

From the definition of  $H$  it is not difficult to realize that such a subsolution  $\bar{W}$  must be on the form

$$\bar{W}(t, x) = -2 \int_0^x \alpha(y) dy + g(t) + K,$$

for some function  $g$  and constant  $K$ . Let  $A(x) = \int_0^x \alpha(y) dy$  and consider only functions  $g$  of the form  $g(t) = 2ct$ , for some constant  $c$ . Then,

$$\bar{W}_t(t, x) = 2c, \quad D\bar{W}(t, x) = -2\alpha(x).$$

For  $\bar{W}$  to be a subsolution,  $A$ ,  $g$  and  $K$  must be chosen so that conditions (5.3)-(5.4) are satisfied. For (5.3) to hold  $\alpha$  must be such that

$$0 \leq \bar{W}_t(t, x) - 2H \left( x, -\frac{D\bar{W}(t, x)}{2} \right) = 2c - 2\lambda(x) \left( e^{\alpha(x)} - 1 \right),$$

which implies that  $\alpha(x)$  must satisfy

$$\alpha(x) \leq \log \left( 1 + \frac{c}{\lambda(x)} \right).$$

By setting  $\alpha(x)$  equal to the right-hand side, equality is achieved in (5.3). Note that it is only for  $c \geq -\inf_{x \leq z} \lambda(z)$  for which  $\alpha$  is guaranteed to be well-defined. For this particular choice of  $\bar{W}$  the terminal condition (5.4) becomes

$$0 \geq \bar{W}(T, z) = 2cT - 2A(z) + K,$$

and the constant  $K$  must satisfy

$$K \leq 2A(z) - 2cT.$$

From the discussion in Section 4 and at the beginning of this section, it is clear that it is desirable to have the initial value  $\bar{W}(0, 0)$  as large as possible. Here,  $\bar{W}(0, 0) = K$  and the inequality gives the upper bound  $2A(z) - 2cT$ ; take  $K$  to equal this upper bound. The resulting subsolution  $\bar{W}$  is given by

$$\bar{W}(t, x) = 2 \int_x^z \log \left( 1 + \frac{c}{\lambda(y)} \right) dy - 2c(T - t).$$

Lastly, the constant  $c$  can now be chosen so as to maximize  $\bar{W}(0, 0)$ : Take  $c = c^*$ ,

$$c^* = \operatorname{argmax} \bar{W}(0, 0) = \operatorname{argmax} \left\{ 2 \int_0^z \log \left( 1 + \frac{c}{\lambda(y)} \right) dy - 2cT \right\},$$

where only  $c > -\inf_{x \leq z} \lambda(x)$  are considered. Differentiability with respect to  $c$  implies that the optimal  $c^*$  must be a solution to the equation

$$\int_0^z \frac{dy}{\lambda(y) + c} = T. \quad (5.7)$$

For this choice of  $c^*$  the subsolution  $\bar{W}$  has initial value

$$\bar{W}(0, 0) = 2 \int_0^z \log \left( 1 + \frac{c^*}{\lambda(y)} \right) dy - 2c^*T.$$

To evaluate the performance of the corresponding importance sampler the initial value  $\bar{W}(0, 0)$  should be compared to  $2U(0, 0)$ , with  $U$  as in (3.2). To this end, we have the following result which suggests asymptotic optimality (rigorously proved in Section 6).

**Proposition 5.1.** *For  $d = 1$ , the large deviation rate is given by*

$$U(0, 0) = \int_0^z \log \left( 1 + \frac{c}{\lambda(y)} \right) dy - cT,$$

where  $c$  solves the equation

$$\int_0^z \frac{dy}{\lambda(y) + c} = T.$$

It is possible to show Proposition 5.1 by means of convex optimization arguments. However, it is a special case of the more general result presented next and a separate proof is therefore omitted.

The following is an excerpt from [6] adapted to the current specific setting. For  $c \in \mathbb{R}$  and  $x, y \in \mathbb{R}^d$ , the *Mañé potential at level  $c$* , denoted by  $S^c(x, y)$ , is defined as

$$S^c(x, y) = \inf \left\{ \int_0^\tau \left( c + L(\psi(s), \dot{\psi}(s)) \right) ds, \psi(0) = x, \psi(\tau) = y \right\},$$

where  $L$  is the local rate function defined in Section 3 and the infimum is taken over  $\psi \in \mathcal{AC}([0, \infty): \mathbb{R}^d)$  and  $\tau > 0$ .

In the current setting, because  $X^n(0) = 0$  (no defaults at time 0), the initial value of interest is  $x_0 = 0$ . The following result shows how  $S^c(0, y)$  relates to the Hamiltonian  $H$ .

**Lemma 5.2** (cf. Proposition 2.1 in [6]). *The function  $y \mapsto S^c(0, y)$  is a viscosity solution of the stationary Hamilton-Jacobi equation*

$$H(y, DS(y)) = c,$$

for all  $y \neq 0$  and  $c > c_H$ , where  $c_H$  satisfies

$$c_H \geq \sup_x \inf_p H(x, p).$$

The constant  $c_H$  is known as *Mañé's critical value*, see [6]. For the specific model considered here (see Example 2.1 in [6]),

$$c_H = - \inf_{x \in \Omega \setminus D_z} \sum_{j=1}^d \lambda_j(x).$$

Note that this is consistent with the discussion leading up to Proposition 5.1, where it was necessary to take  $c > -\inf_{x \leq z} \lambda(x)$ .

The following theorem is basically a combination of results in [6], adapted to the current setting; a proof is immediately obtained from proofs provided in [6]. It is a generalization to higher dimensions of the seemingly ad-hoc method used in this subsection for the one-dimensional case.

**Theorem 5.3** (cf. Theorem 3.1 and Section 5 in [6]). *Consider the collection of functions  $\bar{U}^{c,y}(\cdot, \cdot)$  defined by*

$$\bar{U}^{c,y}(t, x) = S^c(0, y) - S^c(0, x) - c(T - t),$$

*where  $c > c_H$  and  $y \in \partial D_z$ . If  $y \mapsto S^c(0, y)$  is  $C^1(\Omega)$ , then  $\bar{U}^{c,y}(\cdot, \cdot)$  is a classical subsolution to (5.6). Moreover, if  $c$  and  $y$  are chosen such that*

$$\bar{U}^{c,y}(0, 0) = \inf_{y \in \partial D_z} \sup_{c > c_H} \{S^c(0, y) - cT\},$$

*and  $d = 1$ , the corresponding  $\bar{U}^{c,y}(0, 0)$  is equal to the large deviation rate,*

$$\bar{U}^{c,y}(0, 0) = U(0, 0).$$

The result together with the preceeding discussion states that, for  $c > c_H$  and  $y \in \partial D_z$ ,  $\bar{W} = 2\bar{U}^{c,y}$  is a subsolution to the Isaacs equation (5.2) and in the one-dimensional setting  $\bar{W}$  attains the maximal initial value  $2U(0, 0)$  if we choose  $c, y$  appropriately. It follows that the corresponding choice of sampling distribution achieves asymptotic optimality; see Section 6 for a rigorous proof.

**Remark 5.4.** It is conjectured in [6] that the representation therein will hold for  $d \geq 2$  as well. That is, if  $c, y$  are chosen as in the second part of Theorem 5.3, then  $\bar{U}^{c,y}(0, 0) = U(0, 0)$  for  $d \geq 2$  as well. This can be observed for specific cases but the method of proof in [6] does not yet cover higher dimensions.

For  $d = 1$ ,  $\partial D_z = \{z\}$  and the Mañé potential is precisely the function

$$S^c(0, x) = \int_0^x \log\left(1 + \frac{c}{\lambda(y)}\right) dy,$$

Thus, the construction of  $\bar{W}$  according to Theorem 5.3 is precisely the subsolution constructed in a seemingly ad hoc way at the beginning of this section, and Proposition 5.1 becomes a corollary to Theorem 5.3.

**5.3. Sampling distribution for multi-dimensional credit risk model.** We now approach the task of finding an explicit change of measure for the specific credit risk model suggested in [2] and defined in (2.1).

It should be emphasized that Theorem 5.3 and the forthcoming results on performance (Section 6) do not depend on the explicit form of the original jump intensity, described by  $\lambda$ , and hold for any Markovian birth process with the structure described in Section 2. However, the particular choice of  $\lambda$  becomes crucial when computing the explicit change of measure for a specific model, which in the context of Theorem 5.3 amounts to computing the Mañé potential.

For  $d = 1$  the derivation of  $S^c(0, y)$  is provided in Section 5.2. The corresponding sampling distribution has jump intensity  $\bar{\lambda}(x) = \lambda(x)e^{\alpha(x;c)}$ , where

$$\alpha(x, c) = \log\left(1 + \frac{c}{\lambda(x)}\right),$$

and  $c$  solves (5.7). Hence, for the one-dimensional model the change of measure used for importance sampling is completely known up to the constant  $c$ , which one might need to determine numerically.

Now, consider  $d \geq 2$ . From the point of view of Theorem 5.3 the task is to find  $D\bar{W}(t, x) = -2DS^c(0, x)$ . An obvious approach is to try and solve the variational problem in the definition of  $S^c(0, x)$ . Another approach is the one (implicitly) used for  $d = 1$ : Find a function  $\alpha(x; c)$  that solves the equation

$$c = H(x, \alpha(x, c)) = \sum_{j=1}^d \lambda_j(x) \left( e^{\langle \alpha(x, c), e_j \rangle} - 1 \right), \quad (5.8)$$

and then try to find a potential  $A(x; c)$  such that  $DA(x; c) = \alpha(x; c)$ . However, for  $d > 1$ , for there to exist a potential  $A(x; c)$  that has  $\alpha(x; c)$  as its gradient,  $\alpha(x; c)$  must form a conservative vector field. Finding such solutions to (5.8) clearly depends entirely on the choice of  $\lambda$  and appears to be a non-trivial task already for rather simple choices.

Before discussing further the problem of finding efficient sampling distributions for  $d > 1$ , we consider a special case of the credit risk model. Recall that the general form of the jump intensities are

$$\lambda_j(x) = a_j(w_j - x_j)e^{b \sum_{i=1}^d x_i}, \quad j = 1, \dots, d,$$

for some non-negative  $a_j$ 's and  $b$ . When  $a_j = a$  for some  $a \in \mathbb{R}$  and all  $j = 1, \dots, d$ , the model is reduced to the one-dimensional case - all groups are homogeneous and thus can be described as only one group - and the change of measure can once again be found explicitly. Indeed, choose  $\alpha(x; c)$  according to

$$\langle \alpha(x; c), e_j \rangle = \log \left( 1 + \frac{c}{\sum_{i=1}^d \lambda_i(x)} \right), \quad j = 1, \dots, d. \quad (5.9)$$

This defines a conservative vector field and the corresponding potential  $A(x; c)$ , as well as the optimal  $c$ , is analogous to before, with

$$A(x; c) = \int_0^{\sum x_i} \log \left( 1 + \frac{c}{\lambda(y)} \right) dy.$$

Note that this relies on the form of  $\lambda$ , specifically the fact that  $\sum_i \lambda_i(x)$  is a function of  $x_1, \dots, x_d$  only through the sum  $\sum_i x_i$ . An interesting observation is that this choice of sampling distribution amounts to  $D\bar{W}$  being perpendicular to the barrier the process is trying to cross, which seems intuitively appealing.

The choice of  $\alpha(x, c)$  according to (5.9) is a solution to the stationary equation (5.8) for the case of general  $a_i$ 's in (2.1), not just the effectively one-dimensional case discussed in the last paragraph. Thus, in accordance with the previous discussion it is tempting to base the sampling distribution on this choice. However, this  $\alpha(x, c)$  is not a conservative vector field in general. Indeed, for  $d = 2$  the (scalar) curl of  $\alpha(x, c)$  is such that the necessary condition for there to exist a potential  $A(x, c)$  with  $\alpha(x, c) = DA(x, c)$  becomes

$$\frac{e^{b(x_1+x_2)}}{(\lambda_1(x) + \lambda_2(x))^2 + \lambda_1(x) + \lambda_2(x)} (a_1 - a_2) = 0,$$

which clearly does not hold when the two groups have different intensities. At the moment, for general choices of  $a \in \mathbb{R}^d$  and  $b \geq 0$ , it is not known to the authors how to find the Mañé potential  $S^c(0, x)$  corresponding to (2.1); for certain intensity models  $\lambda$  the above choice may still work. Still, the obtained results can be used to guide the design of sampling algorithms and one suggestion is discussed next.

Consider the general form of the credit risk model, where the  $a_j$ 's are non-negative and not necessarily equal. Let  $a^* = \bigvee_{j=1}^d a_j$  and define the vector  $\alpha(x; c)$  by

$$\langle \alpha(x; c), e_j \rangle = \log \left( 1 + \frac{c}{\sum_{j=1}^d a^*(w_j - x_j) e^{b \sum x_i}} \right), \quad j = 1, \dots, d.$$

This choice of  $\alpha(x; c)$  satisfies

$$H(x, \alpha(x; c)) = c \left( \frac{\sum_{j=1}^d a_j (w_j - x_j)}{\sum_{j=1}^d a^*(w_j - x_j)} \right) \leq c,$$

and thus

$$c - H(x, \alpha(x; c)) \geq 0,$$

which is the right inequality for a subsolution to the stationary Hamilton-Jacobi equation. Let  $A(x; c)$  be the potential for the vector field  $\alpha(x; c)$  and define the corresponding  $\bar{W}(t, x)$  by

$$\bar{W}(t, x) = 2A(y; c) - 2A(x; c) - 2c(T - t).$$

This is indeed a subsolution to (5.2) and for the special case with all groups homogeneous  $\bar{W}$  coincides with the optimal subsolution. However, good performance is no longer guaranteed by Theorem 5.3, indeed that result is now used only as a guide in the construction of  $\bar{W}$ . In lieu of theoretical results on performance for this particular choice, the algorithm is studied numerically in Section 6, exhibiting good performance in the rare-event setting.

To determine the constant  $c$ , the physical interpretation of  $c$  as the energy level added to the system can be used. The choice of  $c$  should then be such that the trajectories take the appropriate amount of time reach  $D_z$  and to find this energy level does not add any significant extra computational cost. However, good performance is no longer suggested by Theorem 5.3. In lieu of theoretical results on performance, this algorithm is studied numerically in Section 7 for  $d = 2$  and a particular choice of parameter values.

## 6. PERFORMANCE OF SAMPLING ALGORITHMS

In this section performance of the sampling algorithms of Section 5 is discussed. Specifically, it is shown that performance of a sampling algorithm, as measured by the relative error, based on a subsolution  $\bar{W}$  to (5.2) is determined by the initial value of  $\bar{W}$ . It follows that an algorithm achieves asymptotic optimality if  $\bar{W}(0, 0) = 2U(0, 0)$ . Note that although this seems to be implied by the derivation of the Isaacs equation, the derivation (provided in the Appendix) is only of a formal nature and a rigorous proof is indeed needed.

**Theorem 6.1.** *Let  $\bar{W}$  be a subsolution of (5.2) which is  $C^1(\Omega)$  and an affine function of  $t$ . If  $\hat{p}_n$  is the importance sampling estimator based on the vector of jump intensities  $\bar{\lambda}$  defined in (5.5), then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\bar{\Theta}^n} [\hat{p}_n^2] \leq -\frac{W(0, 0)}{2} - U(0, 0).$$

*Proof.* The likelihood ratio between the sampling distribution  $\bar{\mathbb{Q}}^n$  (corresponding stochastic kernel  $\bar{\Theta}^n$ ) and the original distribution  $\mathbb{P}$  (stochastic kernel  $\Theta^n$ ) can be expressed as

$$\frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} = \exp \left\{ \int_0^{T_{N^z}} (\bar{R}(X^n(s)) - R(X^n(s))) ds + \sum_{k=1}^{N^z} \log \frac{r^n(X^n(T_{k-1}), v_k)}{\bar{r}^n(X^n(T_{k-1}), v_k)} \right\}.$$

To analyze the expectation of the likelihood ratio, define the measure  $m^n(\cdot, \cdot)$  on  $\mathbb{R}^d$ , given  $\mathbb{R}^d$ , by

$$m^n(x, dy) = \sum_{j=1}^d n \lambda_j(x) \delta_{e_j}(dy), \quad x \in \mathbb{R}^d.$$

Furthermore, let  $\{M^n(t, \cdot)\}_t$  denote the point process defined by the jumps of  $X^n$ . That is,  $M^n(t, B)$  is the number of jumps of  $X^n$  in  $(0, t]$  in directions that are in  $B \subset \mathbb{R}^d$ ,

$$M^n(t, B) = n \sum_{j=1}^d X_j^n(t) I\{e_j \in B\}.$$

At any time  $t$ , the instantaneous jump intensity of  $M^n$  is  $m^n(X^n(t), \cdot)$ .

With the jump intensities  $\bar{\lambda}$  taken as in (5.5), the likelihood ratio can be expressed as

$$\begin{aligned} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} = \exp \left\{ \int_0^{T_{N^z}} n H \left( X^n(s), \frac{-D\bar{W}(s, X^n(s))}{2} \right) ds \right. \\ \left. + \frac{1}{2} \sum_{k=1}^{N^z} \langle D\bar{W}(T_{k-1}, X^n(T_{k-1})), v_k \rangle \right\}. \end{aligned}$$

Moreover, from the definitions of  $m^n$  and  $M^n$ , the likelihood ratio takes the form

$$\begin{aligned} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} = \exp \left\{ \int_0^{T_{N^z}} n H \left( X^n(s), \frac{-D\bar{W}(s, X^n(s))}{2} \right) ds \right. \\ + \frac{n}{2} \int_0^{T_{N^z}} \int_{\mathbb{R}^d} \left( \bar{W}(s, X^n(s) + \frac{y}{n}) - \bar{W}(s, X^n(s)) \right) dM^n(s, y) \\ - \frac{n}{2} \sum_{k=1}^{N^z} (\bar{W}(T_{k-1}, X^n(T_{k-1}) + v_k/n) - \bar{W}(T_{k-1}, X^n(T_{k-1}))) \\ \left. + \frac{n}{2} \sum_{k=1}^{N^z} \langle D\bar{W}(T_{k-1}, X^n(T_{k-1})), v_k/n \rangle \right\}. \end{aligned}$$

By partial integration

$$\begin{aligned} \int_0^{T_{N^z}} \int_{\mathbb{R}^d} (\bar{W}(s, X^n(s) + y/n) - \bar{W}(s, X^n(s))) dM^n(s, y) \\ = \bar{W}(T_{N^z}, X^n(T_{N^z})) - \bar{W}(0, 0) - \int_0^{T_{N^z}} \bar{W}_t(s, X^n(s)) ds. \end{aligned}$$

Since  $\bar{W}$  is assumed to be  $C^1(\Omega)$  and the state space is a compact subset of  $\mathbb{R}^d$ , the convergence

$$n \left( \bar{W} \left( t, x + \frac{e_j}{n} \right) - W(t, x) \right) \rightarrow \langle D\bar{W}(t, x), e_j \rangle,$$

as  $n \rightarrow \infty$ , is uniform in  $x$ . Hence, there is a sequence  $C_n$  such that  $C_n \rightarrow 0$  as  $n \rightarrow \infty$ , and

$$\sup_{x \in \Omega, j \in \{1, \dots, d\}} \left| \langle D\bar{W}(t, x), e_j \rangle - n \left( \bar{W} \left( t, x + \frac{e_j}{n} \right) - W(t, x) \right) \right| \leq C_n.$$

The uniform convergence thus implies the upper bound

$$\begin{aligned} N^z C_n &\geq n \sum_{k=1}^{N^z} \left( \langle D\bar{W}(T_{k-1}, X^n(T_{k-1})), v_k/n \rangle \right. \\ &\quad \left. - \bar{W}(T_{k-1}, X^n(T_{k-1}) + v_k/n) + \bar{W}(T_{k-1}, X^n(T_{k-1})) \right), \end{aligned}$$

which gives an upper bound for the likelihood ratio,

$$\begin{aligned} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} &\leq \exp \left\{ -\frac{n}{2} \int_0^{T_{N^z}} \left( \bar{W}_t(s, X^n(s)) - 2H \left( X^n(s), \frac{-D\bar{W}(s, X^n(s))}{2} \right) \right) ds \right. \\ &\quad \left. + \frac{n}{2} \bar{W}(T_{N^z}, X^n(T_{N^z})) - \frac{n}{2} \bar{W}(0, 0) + \frac{1}{2} N^z C_n \right\}. \end{aligned}$$

The assumption that  $\bar{W}$  is a subsolution to (5.2) implies that the first integral is bounded from below by 0. Moreover, by the definition of  $N^z$ ,  $\bar{W}(T_{N^z}, X^n(T_{N^z})) \leq 0$ . Hence, the following upper bound holds for  $\hat{p}_n = I\{N^z < N^0\}(d\mathbb{P}/d\bar{\mathbb{Q}}^n)$ ,

$$\begin{aligned} \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} \right] &\leq \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} \exp \left\{ -\frac{n}{2} \bar{W}(0, 0) + \frac{1}{2} N^z C_n \right\} \right] \\ &= e^{-\frac{n}{2} \bar{W}(0, 0)} \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} e^{\frac{1}{2} N^z C_n} \right]. \end{aligned}$$

The process  $Q^n$ , hence the process  $X^n$ , is defined so that the maximum number of jumps is  $n$ ;  $N^z \leq n$ . Combined with the upper bound just derived for the expectation of  $\hat{p}_n$ , this yields the upper bound

$$\frac{1}{n} \log \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} \right] \leq -\frac{1}{2} \bar{W}(0, 0) + \frac{C_n}{2} + \frac{1}{n} \log p_n.$$

The result now follows from the large deviation principle for  $p_n$  and the definition of the sequence  $C_n$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} \frac{d\mathbb{P}}{d\bar{\mathbb{Q}}^n} \right] \leq -\frac{1}{2} W(0, 0) - U(0, 0).$$

□

Combining Theorem 6.1 and Proposition 5.1 (or Theorem 5.3) we immediately obtain that the optimal time-homogeneous sampling distribution described in Section 5.2, and more generally in Theorem 5.3, achieves asymptotic optimality.

**Corollary 6.2.** *For  $d = 1$ , taking  $\bar{W} = 2\bar{U}^{c,y}$  as in Theorem 5.3 ensures asymptotic optimality.*



## 7. NUMERICAL EXPERIMENTS

In this section we present some numerical experiments for the importance sampling algorithms proposed in Section 5 for different choices of parameter values in (2.1). In particular, we implement the optimal time-homogeneous importance sampler (defined in Theorem 5.3) for the examples studied in [2], verifying numerically the asymptotic optimality of Theorem 6.1.

The results presented in this section are based on the embedded discrete-time Markov chain  $\{X^n(T_j); j \geq 0\}$ , not the continuous-time process  $\{X^n(t); t \in [0, T]\}$ . This has no effect on the theoretical results since the second moment of the estimator based on the discrete-time chain is always lower than the second moment of  $\hat{p}_n$ .

In all simulations the number of obligors is  $n = 125$  and the time of maturity is  $T = 5$ . Moreover, all estimates are based on 100 batches with  $N = 5000$  samples in each batch; probability estimates and relative errors are computed over batches.

Tables 1 and 2 present simulation results for the homogeneous version of the model, with  $a_j = 0.01$ , for all  $j$ , and  $b = 0$  (Table 1) or  $b = 5$  (Table 2). The subsolution used to construct the sampling distribution is precisely that of Theorem 5.3. Recall from Section 5 that the homogeneous version of the model can always be reduced to the one-dimensional setting, regardless of the dimension  $d$ , and the accuracy of a simulation algorithm is not affected; asymptotical optimality is achieved (trivial).

Table 1 corresponds to the example of independent obligors ( $b = 0$ ) studied in Section 4 in [2]. The results in Table 2 are for a model with moderate contagion, in-between the cases of independent obligors ( $b = 0$ ) and extreme contagion ( $b = 13$ ). The choice of parameter values are based on [2] in order to make comparison of the different importance sampling algorithms possible.

TABLE 1. Importance sampling and Monte Carlo estimates for the case of independent obligors;  $a = 0.01$ ,  $b = 0$ .

$z$	Importance sampling		Monte Carlo	
	Estimate	Relative error	Estimate	Relative error
0.10	8.238e-3	0.0219	8.282e-3	0.1389
0.15	1.089e-5	0.027	1.200e-5	3.978
0.20	1.737e-9	0.028	-	-
0.25	7.250e-15	0.031	-	-
0.30	3.499e-20	0.039	-	-
0.35	4.470e-26	0.038	-	-
0.40	1.624e-32	0.037	-	-

In [2] the authors remark that there is little to no need for variance reduction in the presence of extreme contagion, i.e, large values for  $b$ . While this is indeed true, even in models with moderate contagion there appears to be a need for variance reduction, as illustrated by Table 2.

Table 3 shows simulation results for the inhomogeneous model using the subsolution  $\bar{W}$  described in Section 5.3. The results are for two groups ( $d = 2$ ), one constituting 80 percent of the population and having an individual intensity  $a_1 = 0.01$ ,

TABLE 2. Importance sampling and Monte Carlo estimates for a model with moderate contagion;  $a = 0.01$ ,  $b = 5$ .

$z$	Importance sampling		Monte Carlo	
	Estimate	Relative error	Estimate	Relative error
0.10	4.389e-2	0.0183	4.383e-2	0.0613
0.15	9.337e-4	0.0210	9.660e-4	0.480
0.20	9.183e-6	0.0266	1.800e-5	3.196
0.25	2.552e-8	0.0273	-	-
0.30	1.380e-10	0.0295	-	-
0.35	7.280e-13	0.0343	-	-
0.40	4.089e-15	0.0322	-	-

the members of the second group having an individual intensity  $a_2 = 0.05$ . The contagion parameter is  $b = 5$ .

TABLE 3. Importance sampling, using  $\bar{W}$ , and Monte Carlo estimates for inhomogeneous groups;  $d = 2$ ,  $a = (0.01, 0.05)$ ,  $b = 5$ ,  $w = (0.8, 0.2)$ .

$z$	Importance sampling		Monte Carlo	
	Estimate	Relative error	Estimate	Relative error
0.10	0.377	0.0149	0.376	0.0174
0.15	3.118e-2	0.0186	3.105e-2	0.0778
0.20	6.252e-4	0.0283	6.160 e-4	0.570
0.25	1.677e-6	0.0596	-	-
0.30	4.662e-9	0.126	-	-
0.35	7.888e-12	0.261	-	-
0.40	9.756e-15	0.637	-	-

The difference between the homogeneous case, for which the algorithm is asymptotically optimal, and the example of two different groups is apparent by comparison of the results in Tables 2 and 3. For the homogeneous case, the relative error is only a few percent (even for probabilities of order  $10^{-32}$ ), whereas for the inhomogeneous case the relative error increases more rapidly as the probability becomes smaller. This decrease in performance is to be expected since the subsolution  $\bar{W}$  is not the optimal one. However, performance remains good and the algorithm shows substantial improvement compared to standard Monte Carlo; even for probabilities of order  $10^{-15}$  the observed relative error is below 2/3. Comparing with the importance sampling algorithms in [2], implicitly based on affine subsolutions, this illustrates how the subsolution approach can provide significant improvement in the design of efficient algorithms compared to a “naive” change-of-measure.

#### APPENDIX A. DERIVATION OF THE ISAACS EQUATION

For completeness, and for the reader who wishes to develop some intuition, we now proceed with a formal derivation of the Isaacs equation associated with the importance sampling estimator (4.2). Naturally, the argument follows closely the general steps used in other works on the subsolution approach for dynamic

importance sampling; see [12] for an overview. We emphasize that the derivation is of a formal nature and not all steps are motivated rigorously.

Recalling the discussion in Section 5, the quantities of interest are

$$V^n = \inf_{\bar{\Theta}^n} \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} \prod_{k=1}^{N^z} \frac{\Theta^n(d\tau_k, v_k \mid X^n(T_{k-1}))}{\bar{\Theta}^n(d\tau_k, v_k \mid X^n(T_{k-1}))} \right],$$

an associated large deviation type scaling,

$$W^n = -\frac{1}{n} \log V^n,$$

and their time-and-state-dependent analogues

$$V^n(t, x) = \inf_{\bar{\Theta}^n} \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} \prod_{k=l}^{N^z} \frac{\Theta^n(d\tau_k, v_k \mid X^n(T_{k-1}))}{\bar{\Theta}^n(d\tau_k, v_k \mid X^n(T_{k-1}))} \mid X^n(t) = x \right], \quad (\text{A.1})$$

where  $l$  is such that  $T_{l-1} \leq t < T_l$ , and  $W^n(t, x)$  (defined from  $V^n(t, x)$  as  $W^n$  from  $V^n$ ). By the memoryless property of the exponential distribution no correction for the first step is needed in (A.1);  $\tau_l$  and  $\tau_l \mid \tau_l \geq t - T_{l-1}$  have the same distribution.

As described in Section 4 the jump intensities under consideration are of the form  $\bar{r}^n(x, e_j) = n\bar{\lambda}_j(x)$ , where  $\bar{\lambda}(x) = (\bar{\lambda}_1(x), \dots, \bar{\lambda}_d(x))$  is not identical to the zero element in  $\mathbb{R}^d$ . Since each stochastic kernel  $\bar{\Theta}^n$  is determined by the corresponding  $\bar{\lambda}$ , the infimum in (A.1) is over those  $\bar{\lambda}(x)$  that are zero only for directions  $j$  for which  $\lambda$  is zero. Note that there will be a slight abuse of notation in that supremum and infimum is taken over  $\bar{r}^n$ , and in including the argument  $x$  although the optimization will always take place for a fixed state  $x$ .

Because the process  $X^n$  is constant between jumps, the likelihood ratio in (5.1) can be expressed as

$$\begin{aligned} & \exp \left\{ \int_{T_l}^{T_{N^z}} (\bar{R}(X^n(s)) - R(X^n(s))) ds + \sum_{k=l}^{N^z} \log \frac{r^n(X^n(T_{k-1}), v_k)}{\bar{r}^n(X^n(T_{k-1}), v_k)} \right\} \\ &= \exp \left\{ \sum_{k=l}^{N^z} \left( (\bar{R}(X^n(T_{k-1})) - R(X^n(T_{k-1}))) \tau_k \right. \right. \\ & \quad \left. \left. + \log \frac{r^n(X^n(T_{k-1}), v_k)}{\bar{r}^n(X^n(T_{k-1}), v_k)} \right) \right\}. \end{aligned}$$

Integration over the joint distribution of  $(\tau_l, v_l)$ , using the independence of the  $\tau_k$ 's, yields

$$\begin{aligned} V^n(t, x) &= \inf_{\bar{r}^n(x, \cdot)} \int_0^\infty \sum_{j=1}^d \exp \left\{ (\bar{R}(x) - R(x))u + \log \frac{r^n(x, e_j)}{\bar{r}^n(x, e_j)} \right\} \\ & \quad \times \mathbb{E}_{\Theta^n} \left[ I\{N^z < N^0\} \exp \left\{ \int_{T_{l+1}}^{T_{N^z}} (\bar{R}(X^n(s)) - R(X^n(s))) ds \right. \right. \\ & \quad \left. \left. + \sum_{k=l+1}^{N^z} \log \frac{r^n(X^n(T_{k-1}), v_k)}{\bar{r}^n(X^n(T_{k-1}), v_k)} \right\} \mid X^n(t+u) = x + \frac{e_j}{n} \right] \\ & \quad \times r^n(x, e_j) e^{-R(x)u} du, \end{aligned}$$

and by the dynamic programming principle  $V^n(t, x)$  satisfies the following dynamic programming equation,

$$V^n(t, x) = \inf_{\bar{r}^n(x, \cdot)} \left\{ \int_0^\infty \sum_{j=1}^d e^{(\bar{R}(x) - R(x))u} \frac{r^n(x, e_j)}{\bar{r}^n(x, e_j)} \right. \\ \left. \times V^n(t + u, x + \frac{e_j}{n}) r^n(x, e_j) e^{-R(x)u} du \right\}. \quad (\text{A.2})$$

From equation (A.2) for  $V^n(t, x)$  we obtain the corresponding equation for  $W^n(t, x)$ ,

$$nW^n(t, x) = \sup_{\bar{r}^n(x, \cdot)} \left\{ -\log \int_0^\infty \sum_{j=1}^d e^{(\bar{R}(x) - R(x))u} \frac{r^n(x, e_j)}{\bar{r}^n(x, e_j)} \right. \\ \left. \times e^{-nW^n(t+u, x + \frac{e_j}{n})} r^n(x, e_j) e^{-R(x)u} du \right\}.$$

Let  $\hat{\Theta}^n$  denote the stochastic kernel based on a set of jump intensities  $\hat{r}^n(x, \cdot)$ , where  $\hat{r}^n(x, e_j) = n\hat{\lambda}_j(x)$  for some function  $\hat{\lambda}: \Omega \rightarrow \mathbb{R}^d$ . From the relative entropy representation for exponential integrals (see, e.g., [7, 12]),

$$nW^n(t, x) = \sup_{\bar{r}^n(x, \cdot)} \inf_{\hat{\Theta}^n} \left\{ \mathcal{H}(\hat{\Theta}^n \mid \Theta^n) + \int_0^\infty \sum_{j=1}^d ((R(x) - \bar{R}(x))u \right. \\ \left. + \log \frac{\bar{r}^n(x, e_j)}{r^n(x, e_j)} + nW^n(t + u, x + \frac{e_j}{n})) \hat{r}^n(x, e_j) e^{-\hat{R}(x)u} du \right\},$$

where  $\mathcal{H}$  denotes the relative entropy. The infimum over  $\hat{\Theta}^n$  is equivalent to infimum over jump intensities  $\hat{r}^n(x, \cdot)$  and the likelihood ratio between  $\hat{\Theta}^n$  and  $\Theta^n$  is of the same form as the likelihood ratio between  $\Theta^n$  and  $\Theta^n$ . Writing out the relative entropy term explicitly and moving  $nW^n(t, x)$  to the right-hand side,

$$0 = \sup_{\bar{r}^n(x, \cdot)} \inf_{\hat{r}^n(x, \cdot)} \left\{ \int_0^\infty \sum_{j=1}^d \left( (2R(x) - \bar{R}(x) - \hat{R}(x))u \right. \right. \quad (\text{A.3})$$

$$\left. + \log \bar{r}^n(x, e_j) + \log \hat{r}^n(x, e_j) - 2 \log r^n(x, e_j) \right. \quad (\text{A.4})$$

$$\left. + n(W^n(t + u, x + \frac{e_j}{n}) - W^n(t, x)) \right) \hat{r}^n(x, e_j) e^{-\hat{R}(x)u} du \Big\}. \quad (\text{A.5})$$

The three terms on the right-hand side are treated separately. The first two integrals are straightforward to compute:

$$\int_0^\infty \sum_{j=1}^d \left( 2R(x) - \bar{R}(x) - \hat{R}(x) \right) u \hat{r}^n(x, e_j) e^{-\hat{R}(x)u} du \\ = \frac{1}{\hat{R}(x)} \left( 2R(x) - \bar{R}(x) - \hat{R}(x) \right),$$

and

$$\begin{aligned} & \int_0^\infty \sum_{j=1}^d (\log \bar{r}^n(x, e_j) + \log \hat{r}^n(x, e_j) - 2 \log r^n(x, e_j)) \hat{r}^n(x, e_j) e^{-\hat{R}(x)u} du \\ &= \frac{1}{\hat{R}(x)} \sum_{j=1}^d \hat{r}^n(x, e_j) (\log \bar{r}^n(x, e_j) + \log \hat{r}^n(x, e_j) - 2 \log r^n(x, e_j)). \end{aligned}$$

The third integral, with integrand given in (A.5), can be expressed as an expectation involving an exponentially distributed random variable. Indeed, let  $\{\xi_n\}$  be a sequence of random variables each having an exponential distribution with mean  $\hat{R}(x)^{-1}$ . Then,

$$\begin{aligned} & \int_0^\infty \sum_{j=1}^d n \left( W^n(t+u, x + \frac{e_j}{n}) - W^n(t, x) \right) \hat{r}^n(x, e_j) e^{-\hat{R}(x)u} du \\ &= \sum_{j=1}^d \frac{\hat{r}^n(x, e_j)}{\hat{R}(x)} \mathbb{E} \left[ n(W^n(t + \xi_n, x + \frac{e_j}{n}) - W^n(t, x)) \right]. \end{aligned}$$

The expression involving (the integrals of) (A.3)-(A.5) as

$$\begin{aligned} 0 &= \sup_{\bar{r}^n(x, \cdot)} \inf_{\hat{r}^n(x, \cdot)} \left\{ \frac{1}{\hat{R}(x)} (2R(x) - \bar{R}(x) - \hat{R}(x)) \right. \\ &\quad + \frac{1}{\hat{R}(x)} \sum_{j=1}^d \hat{r}^n(x, e_j) (\log \bar{r}^n(x, e_j) + \log \hat{r}^n(x, e_j) - 2 \log r^n(x, e_j)) \\ &\quad \left. + \sum_{j=1}^d \frac{\hat{r}^n(x, e_j)}{\hat{R}(x)} \mathbb{E} \left[ n(W^n(t + \xi_n, x + \frac{e_j}{n}) - W^n(t, x)) \right] \right\}. \end{aligned}$$

Define the function  $l: \mathbb{R} \rightarrow [0, \infty]$  by

$$l(x) = \begin{cases} x \log x - x + 1, & x \geq 0 \\ \infty, & \text{otherwise.} \end{cases}$$

With  $\xi_n \sim \text{Exp}(n\hat{\Lambda}(x))$  and using the same notation for the jump intensities  $\hat{r}^n$  and  $\bar{r}^n$  (i.e.,  $\hat{\lambda}, \hat{\Lambda}$  and  $\bar{\lambda}, \bar{\Lambda}$ ), the equation of interest can be expressed as

$$\begin{aligned} 0 &= \sup_{\bar{\lambda}(x)} \inf_{\hat{\lambda}(x)} \left\{ \frac{1}{\hat{\Lambda}(x)} \left( 2 \sum_{j=1}^d \lambda_j(x) l\left(\frac{\hat{\lambda}_j(x)}{\lambda_j(x)}\right) - \sum_{j=1}^d \bar{\lambda}_j(x) l\left(\frac{\hat{\lambda}_j(x)}{\bar{\lambda}_j(x)}\right) \right) \right. \\ &\quad \left. + \frac{1}{\hat{\Lambda}(x)} \sum_{j=1}^d \hat{\lambda}_j(x) \mathbb{E} \left[ n(W^n(t + \xi_n, x + \frac{e_j}{n}) - W^n(t, x)) \right] \right\}. \end{aligned}$$

Denote by  $W_t$  and  $DW$  the time derivative of  $W$  and the gradient in the space variable  $x$ , respectively. To formally obtain a limit PDE related to the stochastic control problem, assume that there is a suitable limit  $W$  for  $W^n$ . More precisely, that there is a smooth function  $W$  such that, as  $n \rightarrow \infty$ ,  $W^n(t, x) \rightarrow W(t, x)$  and

$$n(W^n(t + \frac{u}{n}, x + \frac{e_j}{n}) - W^n(t, x)) \rightarrow uW_t(t, x) + \langle DW(t, x), e_j \rangle.$$

Consider the expectation involving the  $\xi_n$ 's. By a change of variable,

$$\begin{aligned} & \mathbb{E} \left[ n(W^n(t + \xi_n, x + \frac{e_j}{n}) - W^n(t, x)) \right] \\ &= \int_0^\infty n \left( W^n(t + \xi, x + \frac{e_j}{n}) - W^n(t, x) \right) n \hat{\Lambda}(x) e^{-n \hat{\Lambda}(x) \xi} d\xi \\ &= \int_0^\infty n \left( W^n(t + \frac{\tau}{n}, x + \frac{e_j}{n}) - W^n(t, x) \right) \hat{\Lambda}(x) e^{-\hat{\Lambda}(x) \tau} d\tau. \end{aligned}$$

As  $n$  goes to infinity, the assumed convergence of  $W^n$  implies that, for each  $\tau$ , the integrand converges to  $\tau W_t(t, x) + \langle DW(t, x), e_j \rangle$ . Taking the limit inside the expectation,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ n(W^n(t + \xi_n, x + \frac{e_j}{n}) - W^n(t, x)) \right] = \frac{W_t(t, x)}{\hat{\Lambda}(x)} + \langle DW(t, x), e_j \rangle.$$

Thus, in the limit as  $n$  goes to infinity, the dynamic programming equation for  $W^n$  gives rise to the following Isaacs equation,

$$\begin{aligned} 0 = \sup_{\bar{\lambda}(x)} \inf_{\hat{\lambda}(x)} \frac{1}{\hat{\Lambda}(x)} & \left\{ \sum_{j=1}^d \left( 2\lambda_j(x) l\left(\frac{\hat{\lambda}_j(x)}{\lambda_j(x)}\right) - \bar{\lambda}_j(x) l\left(\frac{\hat{\lambda}_j(x)}{\lambda_j(x)}\right) \right) + W_t(t, x) \right. \\ & \left. + \sum_{j=1}^d \hat{\lambda}_j(x) \langle DW(t, x), e_j \rangle \right\}. \end{aligned} \quad (\text{A.6})$$

Note that, aside from the time derivative  $W_t(x, t)$ , this is equation (6.4) in [8] (with  $DW(x)$  replaced by  $DW(t, x)$ ). Define the Hamiltonian  $\mathbb{H}$  on  $\Omega \times \mathbb{R}^d$  by

$$\mathbb{H}(x, \alpha) = \sup_{\bar{\lambda}(x)} \inf_{\hat{\lambda}(x)} \left\{ \sum_{j=1}^d \left( 2\lambda_j(x) l\left(\frac{\hat{\lambda}_j(x)}{\lambda_j(x)}\right) - \bar{\lambda}_j(x) l\left(\frac{\hat{\lambda}_j(x)}{\lambda_j(x)}\right) + \hat{\lambda}_j(x) \langle \alpha, e_j \rangle \right) \right\}.$$

Recall the definition (3.1) of the Hamiltonian  $H$ ,

$$H(x, \alpha) = \sum_{j=1}^d \lambda_j(x) \left( e^{\langle \alpha, e_j \rangle} - 1 \right).$$

The following result from [8] characterizes saddle points of  $\mathbb{H}$ .

**Proposition A.1** (Proposition 6.2 in [8]). *For any  $x \in \Omega$  and  $\alpha \in \mathbb{R}^d$ ,*

$$\mathbb{H}(x, \alpha) = -2H(x, -\frac{\alpha}{2}),$$

*and the saddle point for  $\mathbb{H}$  is given by  $(\bar{\lambda}, \hat{\lambda})$  such that*

$$\bar{\lambda}_j(x) = \hat{\lambda}_j(x) = \lambda_j(x) e^{-\frac{\langle \alpha, e_j \rangle}{2}}.$$

As mentioned in [8], from the existence of saddle points for  $\mathbb{H}$  one can argue that the factor  $\hat{\Lambda}(x)^{-1}$  in (A.6) can be removed. Indeed, the time derivative does not change this and the Isaacs equation (A.6) becomes

$$W_t(x, t) + \mathbb{H}(x, DW(x, t)) = 0. \quad (\text{A.7})$$

From the definition of the value function  $V^n(t, x)$  it is clear that  $W^n(t, x)$  must satisfy the terminal condition

$$W^n(T, x) = \begin{cases} 0, & x \in D_z, \\ \infty, & \text{otherwise,} \end{cases}$$

which in turn carries over to the function  $W$ . Proposition A.1 combined with this terminal condition implies that the Isaacs equation (A.7) is indeed the Hamilton-Jacobi equation

$$\begin{cases} W_t(t, x) - 2H\left(x, -\frac{DW(t, x)}{2}\right) = 0 & (t, x) \in [0, T) \times \Omega \setminus D_z, \\ W(T, x) = 0, & x \in D_z. \end{cases} \quad (\text{A.8})$$

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